Prove series: Linear System

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key to symbols

$\mathbb{C}$ means the complex domain

$\dot{x} = \frac{dx}{dt}$

$x \in \mathbb{C}^n$ is a state

$\mathrm{A} \in \mathbb{C}^{\mathrm{nxn}}$ is the transfer matrix

$x(0) = x_0$

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The math behind $\dot{x} = \mathbf{A}x$?

Solution:

We know that any function could be represented by the power series such that(this will be proven at a different paper):

$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 .... + c_n t^n$

such that $c_i \in \mathbb{C}^n$

Also given that $x(0)$ we have

$\mathrm{x}(0) = \mathrm{c}_0$

$\dot{x}(t) = c_1 + 2c_2 t^1 + 3c_3 t^2 + 4c_4 t^3 .... + n c_n t^{n-1}$

So with the expression $\dot{x} = \mathrm{A}x$, we have:

$$c_1 + 2c_2 t^1 + 3c_3 t^2 + 4c_4 t^3 .... + n c_n t^{n-1} = A\left(c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 .... + c_n t^n\right)$$

This could also be written as:

$$c_1 + 2c_2 t^1 + 3c_3 t^2 + 4c_4 t^3 .... + n c_n t^{n-1} \text{ - } \mathrm{Ac}_0 - \mathrm{Ac}_1 t - \mathrm{Ac}_2 t^2 - \mathrm{Ac}_3 t^3 - \mathrm{Ac}_4 t^4 .... - \mathrm{Ac}_n t^n = 0$$

Now you must also take my words that the power series are indpendent. This means that $1, t, t^2, t^3, ..., t^n$ are all independent from each other. Independence means that 1 cannot be used to form the other. (I will also prove this in another paper and define linear independence more clearly) Since they are independent, we can rearrange and collect the terms.

$$(c_1 - \mathrm{Ac}_0) + (2c_2 - \mathrm{Ac}_1)t + (3c_3 - \mathrm{Ac}_2)t^2 + (4c_4 - \mathrm{Ac}_3)t^3 + ...$$

$$........ + (n c_n - \mathrm{Ac}_{n-1})t^{n-1} \text{ - } \mathrm{Ac}_n t^n = 0$$

Since they are independent, each collected terms must also equal to zero

$(c_1 - \mathrm{Ac}_0) = 0$

$(2c_2 - \mathrm{Ac}_1)t = 0$

$(3c_3 - \mathrm{Ac}_2)t^2 = 0$

$$(4c_4 - \mathrm{A}c_3)t^3 \ = 0$$

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$$(nc_n - \mathrm{A}c_{n-1})t^{n-1} = 0$$

$$\mathrm{A}c_n t^n = 0$$

From these equation we can solve for all the constants of c by rearrangement.

$$c_0 = x_0$$

$$c_1 = \mathrm{A}c_0 \ = \ \mathrm{A}x_0$$

$$c_2 = \frac{\mathrm{A}c_1}{2!} \ = \ \frac{A^2x_0}{2!}$$

$$c_3 = \frac{\mathrm{A}c_2}{3} = \frac{A^3x_0}{3!}$$

$$c_4 = \frac{\mathrm{A}c_3}{4} = \frac{A^4x_0}{4!}$$

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$$c_n = \frac{A^nx_0}{n!}$$

Do you remember that in the beginning we have defined x

$$x(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 .... + c_nt^n$$

Now that we know all the constants for c_i, we also know what x is:

$$x(t) = x_0 + Ax_0t + \frac{A^2x_0}{2!}t^2 + \frac{A^3x_0}{3!}t^3 + \frac{A^4x_0}{4!}t^4 .... + \frac{A^nx_0}{n!}t^n$$

If you are a big nerd like me, you might just notice that this is the taylor's series expansion. (This will also be proven in another papger)

$$x(t) \ = e^{At}x_0$$

For most differential equation student, they will notice that the solution is no different from the 1 dimensional solution. This is true, the solution is indeed the same. However, it is important to take note that A in this case is a matrix, not a scalor function.